Meson Production in $p+d$ Collisions and the $I=0$ π-π Interaction. IV. Double-Pion Production and Pion-Pion Scattering*†

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We present an analysis of the measurements on the reaction $p+d\rightarrow He^3+2\pi$, and show that the anomalous bump in the He^3 spectrum cannot be accounted for by $\pi-He^3$ final-state interactions, by Bose statistics for the two pions, or by the deuteron and He^3 wave functions. However, the anomaly can be satisfactorily explained by a strong final-state interaction between the two pions. We determine the S-wave π-π scattering length in the $I=0$ state to be $2\hbar/\mu c$.

I. INTRODUCTION

MEASUREMENTS of momentum spectra of the He^3 produced in high-energy $p+d$ collisions have been extensively discussed in the preceding three papers. Preliminary results of the analysis in terms of theoretical models have also been published.[1,2] In this paper we give a brief review of the earlier analysis and present the results of some new calculations.

II. STATISTICAL MODEL

As a first step in comparing the He^3 and H^3 momentum spectra with theory, we have considered the statistical model—that is, we have calculated what is commonly called phase space. There are two ways to do this—the noninvariant way as used by Fermi,

$$\rho=\int d\mathbf{p}_3 d\mathbf{p}_4 d\mathbf{p}_5 \delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4-\mathbf{p}_5)\times\delta(\omega_1+\omega_2-\omega_3-\omega_4-\omega_5),$$

where the element of volume is $d\mathbf{p}$, and the invariant way,

$$\rho=\int dp_3 dp_4 dp_5 \delta(p_1+p_2-p_3-p_4-p_5)\prod_{i=3}^{5}(p_i^2-m_i^2),$$

where the element of volume is dp (four-vector). Here, we call particles 1 and 2 the incoming proton and deuteron, respectively, particle 3 the He^3 or H^3, and particles 4 and 5 the two pions. The invariant phase space integrates to give

$$\frac{d^2\rho}{dp_3 d\Omega_3}=\frac{p_3^2}{\omega_3}\left(1-\frac{4\mu^2}{w^2}\right)^{1/2} \quad (1)$$

in the laboratory system, where w is the total energy in the barycentric system of the two pions of mass μ. Figure 1 shows a comparison of the types of phase space. They differ somewhat, but for ease of calculation and for aesthetic reasons we will consider only the invariant type.

Figure 2 shows our attempt to fit the He^3 spectra measured in the first run. The dashed curves correspond to Eq. (1) fitted to all the points; the solid curves fit only that part of the data outside the apparent peaks. The phase-space calculations can not reproduce the bumps, but give a reasonable fit to the lower momentum end of each spectrum.

In the second run we measured in detail the H^3 spectrum in the reaction $p+d\rightarrow H^3+\pi^++\pi^0$. This spectrum showed no anomaly but agreed satisfactorily with phase space. Recalling the isotopic-spin relationships written in paper I, we conclude that the anomaly is peculiar to an $I=0$ state. Moreover, we can subtract out the $I=1$ contribution to the He^3 spectra to give the $I=0$ part. We have a complete H^3 spectrum at a laboratory angle of 11.8 deg and a single measurement at 15.7 deg in the middle of the continuum. Knowing the behavior of the H^3 spectrum, we can subtract the $I=1$ contribution from each of the He^3 spectra. Figure 3 shows the results for 11.8 deg and 13.5 deg. In Fig. 4

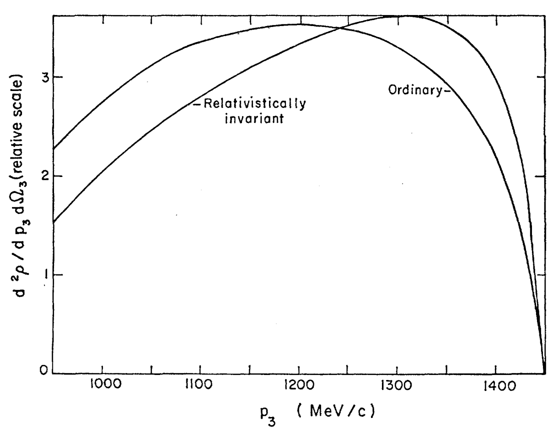

FIG. 1. Comparison of phase-space calculations for the reaction $p+d\rightarrow He^3+2\pi^0$ at an incident proton energy of 743 MeV and a He^3 laboratory angle of 11.7 deg.

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[1] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters 5, 258 (1960).
[2] N. E. Booth, A. Abashian, and K. M. Crowe, Phys. Rev. Letters 7, 35 (1961).

we show how the $I=0$ and $I=1$ contributions depend upon laboratory angle. The $I=1$ contribution varies as a power of the π-π momentum because the pions must be in odd angular-momentum states. We have taken q to be the maximum π-π momentum at each laboratory angle.

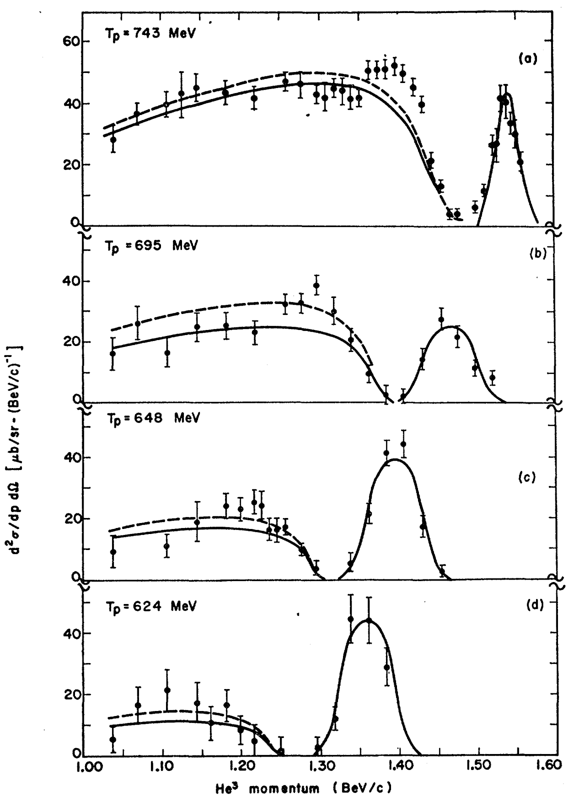


FIG. 2. The He^3 momentum spectra at 11.7 deg in the laboratory system for various proton energies, showing phase-space fits.

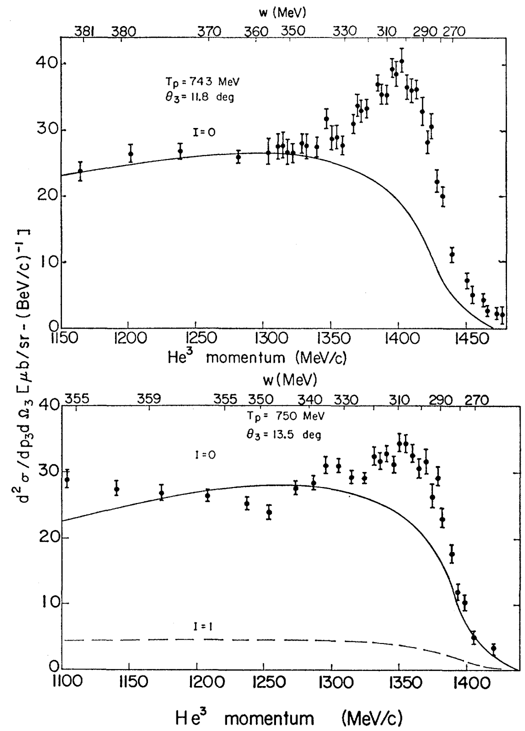


FIG. 3. $I=0$ part of He^3 spectra at 11.8 deg and 13.5 deg. The curves are Eq. (1) with resolution folded in fitted to the low momentum points.

III. WIDTH OF THE ANOMALY

Let us begin this section by assuming that the anomaly is something added to the phase space, for example a particle. In our first communication we showed that the anomaly appeared to behave kinematically as a particle as the incident proton energy was changed.[1] However, the peaks obtained by subtracting the phase space appeared to be broader than the resolution function. Our subsequent measurements and resolution calculations confirm this. Figure 5 shows the results of the subtractions for 11.8 deg and 13.5 deg compared with the computed resolution functions. Unfolding gives a natural linewidth of $\sim$25 MeV, corresponding to a lifetime of $\sim 3\times10^{-23}$ sec. Moreover, we notice a slight but perhaps significant shift of the peak with respect to the w or mass scale. The 15.7-deg data shown in Fig. 6 when phase space is fitted show no particular peaking. Thus, the anomaly does not behave kinematically as a particle and has a lifetime of the same order as the interaction time. We conclude that the anomaly is an enhancement in the two-pion production in the $I=0$ state. We came to the same conclusion independently in paper III. A logical explanation is that we are observing a strongly attractive low-energy pion-pion interaction.

IV. OTHER POSSIBLE EXPLANATIONS FOR THE ANOMALY

Before proceeding further with the analysis in terms of a pion-pion interaction, we consider some other

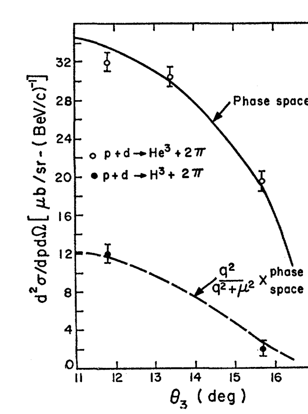


FIG. 4. Differential cross sections for 2π production as a function of laboratory angle of He^3 or H^3. The experimental points are typical of the rather flat region of the spectra. The solid curve is Eq. (1) fitted to the He^3 data. The dashed curve is Eq. (1) multiplied by the factor $q^2/(q^2+\mu^2)$, where q is the π-π relative momentum, to approximate the expectation for P-wave pions.

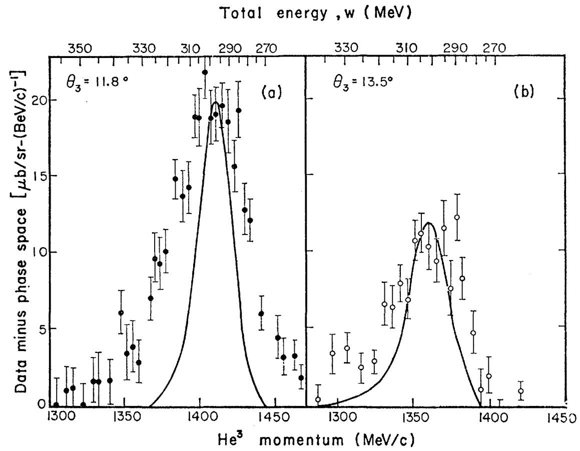


FIG. 5. Subtraction of phase space, fitted as in Fig. 3, from 11.8-deg and 13.5-deg data. The curves are the calculated resolution functions.

factors which have occurred to us or have been suggested by others.[3]

A. π-He^3 Final-State Interaction

We have already presented some arguments[2] why the anomaly could not be due to a final-state interaction between the He^3 and one of the pions. For instance, as the proton energy or angle of observation is changed, the anomaly does not appear at the same relative π-He^3 energy. However, we will now support these arguments with a calculation. A convenient method of calculation is the isobar model of Sternheimer and Lindenbaum.[4] Using this model we assume that the reaction proceeds via the two-step process $p+d \to$ He^{3*} $+\pi^-$ and He$^{3*} \to$ He$^3+\pi^+$. We assume isotropy in the production and decay of the isobar and that in the rest frame of the He^3 each pion energy is weighted by the total π^+-p cross section. We have assumed no broadening of the 3-3 resonance due to internal motion of the He^3. The maximum pion energy in this frame is 200 MeV for the reaction $p+d \to$ He$^3+2\pi$ at $T_p=743$ MeV. An integral over π-He^3 energies is evaluated and the resultant spectrum of the He^3 determined. In Fig. 7 we show a comparison between the isobar model and the statistical model plotted against the total energy in the 2π system. As can be seen, there is no more than a $\pm 10\%$ deviation from ordinary phase space. Moreover, the isobar model gives a dip rather than a peak at $w \sim 300$ MeV. In this calculation we have assumed a sharp 3-3 resonance and that the pions are always involved in it. In other words, the calculation gives the maximum effect to be expected, and more reasonable assumptions would tend to smooth it out.

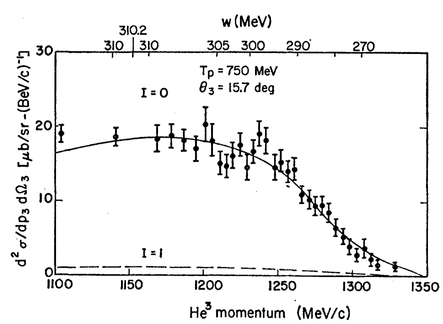


FIG. 6. $I=0$ part of He^3 spectrum at 15.7 deg. The solid curve is Eq. (1) with resolution folded in fitted to all the points.

B. Center-of-Mass Angular Distribution

A tacit assumption in our statistical-model calculations is that the two pions are emitted isotropically in the $p+d$ center-of-mass system. However, most pion-production reactions are strongly anisotropic, e.g., $p+p \to d+\pi^+$, and Selove[3] has pointed out that, since a fixed laboratory angle corresponds to a range of angles in the c.m. system and that 90 deg in the c.m. system appears somewhere in the middle of a momentum spectrum (see Fig. 1 of I), an inhibition of 90 deg production would tend to put a dip in the middle of the spectrum. That is, a depression in the center of the spectrum looks just like a bump at the end. We answer this as follows: Firstly, although the reaction $p+d \to$ He$^3+\pi^0$ is peaked strongly for the He^3 going backwards in the c.m. system, it is rather flat for the He^3 going forwards, and our bump occurs at He^3 angles of 60 to 80 deg. The depression of the forward peaking is due to the deuteron and He^3 wave functions and is discussed in a later section (see also II). The same wave functions occur in the two-pion production, and it would be surprising to see a strong peaking at the forward angles. Secondly, we have divided the 11.8, 13.5, and 15.7 deg data by the phase space and plotted the results against the angle in the c.m. system in Fig. 8. There appears to be little correlation between the three sets of data. Thus,

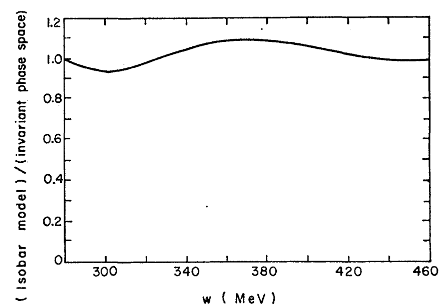


FIG. 7. Comparison of isobar and statistical models for the energy spectrum of He^3 in $p+d \to$ He$^3+2\pi$ at $T_p=743$ MeV. The abscissa is the total energy in the 2π barycentric system.

[3] Discussion following N. E. Booth, A. Abashian, and K. M. Crowe, Rev. Mod. Phys. **33**, 393 (1961).

[4] R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. **109**, 1723 (1958).

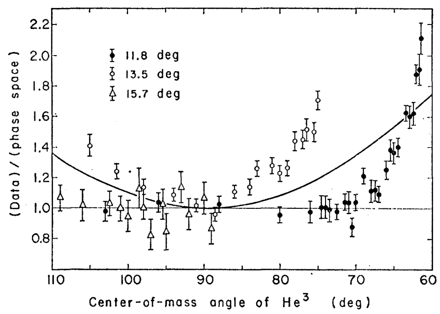


FIG. 8. Plot of the data divided by phase space and normalized to unity at 90 deg versus angle in the c.m. system. Only points that are unaffected by the resolution have been included. The curve shows $(1+3\cos^2\theta^*)$ for comparison.

although such an angular dependence should be considered, it is not in itself an explanation for the anomaly.

C. Bose Effect for the Two Pions

Including the concept of isotopic spin, the two pions in the $I=0$ part of $p+d\rightarrow He^3+2\pi$ are identical bosons. Accordingly, they must have a properly symmetrized wave function, whereas in the statistical-model calculations their wave functions were taken to be independent. The effect of symmetrization is to make the two pions tend to come out together, which is just what we see. Goldhaber *et al.*[5] have shown how to estimate this effect. Unfortunately their calculation introduces a parameter R, the radius of interaction in which the wave function is to be symmetrized. The result of the calculation is that the volume element in phase space is given by

$$\frac{d^2p}{dp_3 d\Omega_3}=\frac{p_3^2}{\omega_3}\left(1-\frac{4\mu^2}{w^2}\right)^{1/2} \times\left\{1+\exp\left[-\left(\frac{R}{2.15}\right)^2(w^2/\mu^2-4)\right]\right\},$$

where R is the radius of interaction in pion units. Figure 9 shows the quantity $\{\ \}$, which we call the Bose factor $B(w)$, plotted versus w for various values of R, and Fig. 10 shows the effect the symmetrization has upon the phase space. For R very large or small there is no effect. A maximum effect is observed for $R\simeq 2\hbar/\mu c$. One might expect a physically reasonable value of R to lie between the range of the π-π force and the radius of the He^3. This effect cannot explain the anomaly, but we do include it later as it significantly changes the shape of the phase space and affects the magnitude of the pion-pion interaction which we deduce from the data.

[5] G. Goldhaber, S. Goldhaber, W. Lee, and A. Pais, Phys. Rev. **120**, 300 (1960).

D. He^3 Form Factor

Because only a fraction (about 10^{-3}) of pion-producing $p+d$ collisions result in the formation of a He^3 or H^3 nucleus, we know that it is improbable that the three nucleons involved stick together to form a bound state. It is worthwhile to look into this nuclear physics "sticking probability" and the factors that control it, because it constitutes the main difference between our experiments and experiments on pion production in pion-nucleon and nucleon-nucleon collisions.

A theoretical framework has been found for treating this situation and has been shown to work satisfactorily in several cases. The theoretical framework is the impulse approximation,[6] and it has been adapted by Ruderman[7] and Bludman[8] to predict angular distributions of the reaction $p+d\rightarrow H^3+\pi^+$. Other authors have used the method to calculate both single and double meson production in similar cases.[9]

In Sec. IV of paper II we outlined how to calculate the form factor $|f(\mathbf{\Delta})|^2$, the probability of forming a final state He^3 or H^3 as a function of $\mathbf{\Delta}$, the momentum transfer to the struck deuteron. To calculate the angular distribution of the reaction $p+d\rightarrow He^3+\pi^0$, one needs to know $|f(\mathbf{\Delta})|^2$ and the angular distribution of the reaction $p+n\rightarrow d+\pi^0$ or, through charge independence, $p+p\rightarrow d+\pi^+$. In a similar way the momentum distribution of He^3 in the reaction $p+d\rightarrow He^3+2\pi$ is given in terms of $|f(\mathbf{\Delta})|^2$ and the angular distribution of the deuteron in the reaction $p+p\rightarrow d+2\pi$. In fact we showed that for $p+d$ collisions at an incident proton

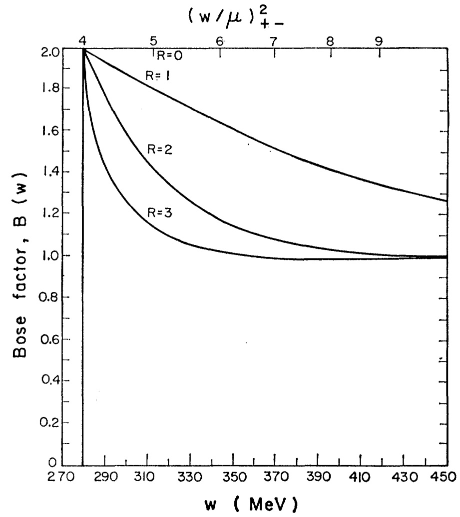


FIG. 9. Bose enhancement factors for various values of the radius of symmetrization R.

[6] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[7] M. A. Ruderman, Phys. Rev. **87**, 383 (1952).
[8] S. A. Bludman, Phys. Rev. **94**, 1722 (1954).
[9] K. R. Greider, Phys. Rev. **122**, 1919 (1961); Yu. K. Akimov, O. V. Savchenko, and L. M. Soroko, Zh. Eksperim. i Teor. Fiz. **41**, 708 (1961) [translation: Soviet Phys.—JETP **14**, 512 (1962)].

energy of 750 MeV, we have

$$(d\sigma/d\Omega)(p+d\rightarrow \text{He}^3+\pi^0) \propto |f(\mathbf{\Delta})|^2(1+\cos^2\theta^*) \quad (2)$$

and

$$(d^2\sigma/dpd\Omega)(p+d\rightarrow \text{He}^3+2\pi) \propto \text{(phase space)}\ |f(\mathbf{\Delta})|^2(1+\cos^2\theta^*), \quad (3)$$

where θ^* is the angle of the heavy particle in the c.m. system. The factor $(1+\cos^2\theta^*)$ is the angular distribution in the reactions $p+p\rightarrow d+\pi^+$ and $p+p\rightarrow d+2\pi$ at proton energies that give the same momenta of the pions in the c.m. system as the corresponding $p+d$ reactions. Figure 11 shows the factors multiplying the phase space in Eq. (3), for the three angles measured in our experiments, plotted against the laboratory momentum of the He^3. Since $|\mathbf{\Delta}|$ is proportional to the laboratory momentum of the He^3, the factor $|f(\mathbf{\Delta})|^2$ decreases monotonically with increasing He^3 momentum. This is counteracted to some extent at the high-momentum end of the phase space by the factor $(1+\cos^2\theta^*)$ and enhanced at the low-momentum end. As can be seen from Fig. 11, the effect of these factors is large, but the procedure by which they are obtained is well-tested, and they must be included in any systematic treatment of the He^3 momentum spectra.

We therefore compute a new phase space, ϕ_f, which is obtained by multiplying the invariant phase space of Eq. (1) by the He^3 form factor effects.

$$\phi_f=\frac{p_3^2}{\omega_3}\left[1-\frac{4\mu^2}{w^2}\right]^{1/2}|f(\mathbf{\Delta})|^2(1+\cos^2\theta^*). \quad (4)$$

V. PION-PION INTERACTION

We have seen how none of the factors considered so far is able to fit the He^3 momentum spectra in the region of the anomaly. The most promising explanation is that the anomaly is due to a strong S-wave π-π interaction that can be characterized by a scattering length and an effective range.[10] According to Watson's theory

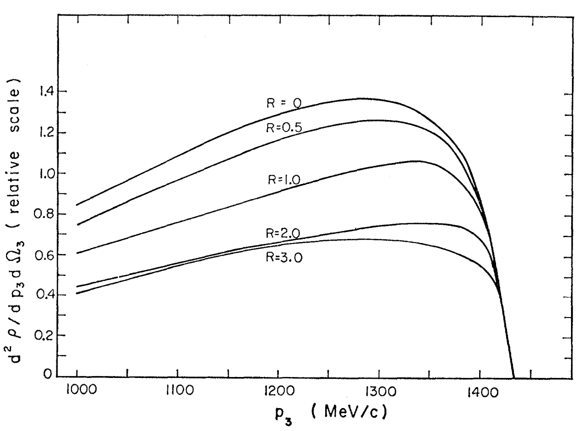


FIG. 10. Effect of Bose enhancement factors on phase space for the reaction $p+d\rightarrow \text{He}^3+2\pi$.

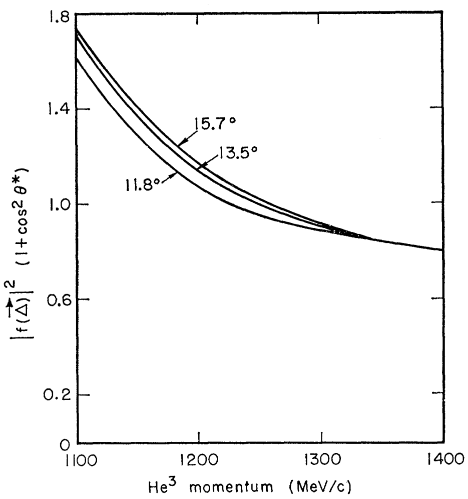


FIG. 11. Plot of the quantity $|f(\mathbf{\Delta})|^2(1+\cos^2\theta^*)$ at three different laboratory angles. The curves are normalized to unity at $\theta^*=90$ deg.

of final-state interactions,[11] the volume element in phase space for a given pion-pion energy w is enhanced by a factor proportional to the pion-pion scattering cross section at the energy w. This is just what one would expect on the basis of "Golden Rule No. 2"—that the transition probability is given by the product of the square of the matrix element and the density of final states.[12] Watson's theory predicts a particular form for the matrix element. The validity conditions of Watson's theory of final-state interactions are that the mechanism of the primary reaction be a short-range interaction, that the final-state interaction be strong and attractive, and that we consider only low relative energies of the two pions. All these conditions are satisfied here.

The usual effective range formula,

$$q\cot\delta=(1/a_{s0})+\tfrac{1}{2}r_0q^2,$$

where δ is the π-π phase shift, a_{s0} the scattering length, and r_0 the effective range, is not applicable here because pions become relativistic very rapidly as the energy increases. We choose to use the formula

$$\left(\frac{q^2}{q^2+\mu^2}\right)^{1/2}\cot\delta=\frac{1}{a_{s0}}+\frac{2}{\pi}\left(\frac{q^2}{q^2+\mu^2}\right)^{1/2}\ln[q+(q^2+\mu^2)^{\frac{1}{2}}] \quad (5)$$

which comes from the S-dominant solutions of the π-π equations of Chew and Mandelstam.[13] That this formula is still a very good approximation for solutions with a P-wave π-π resonance has been pointed out by Desai[14] and by Jackson and Kane.[15] The π-π enhance-

[10] T. N. Truong, Phys. Rev. Letters **5**, 308 (1961).
[11] K. M. Watson, Phys. Rev. **88**, 1163 (1952).
[12] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950), p. 142.
[13] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[14] B. Desai, Phys. Rev. Letters **6**, 497 (1961).
[15] J. D. Jackson and G. L. Kane, Nuovo Cimento **23**, 444 (1962).

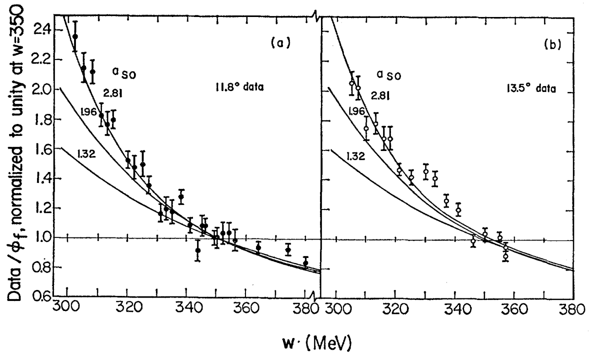


FIG. 12. Plots of 11.8-deg and 13.5-deg data divided by ϕ_f = (phase space) $|f(\mathbf{\Delta})|^2(1+\cos^2\theta^*)$, i.e., right-hand side of Eq. (3). The solid curves are Desai's π-π enhancement factors[14] normalized to unity at $w=350$ MeV.

ment factor is then

$$F(q^2)=(q^2+\mu^2)(\sin^2\delta/q^2),$$

where δ is given by Eq. (5). Figure 12 shows some of these enhancement factors as computed by Desai normalized to unity at $w=350$ MeV $[w=2(q^2+\mu^2)^{\frac{1}{2}}]$. Also shown are the ratios of data to the modified phase space ϕ_f given by Eq. (4) for the 11.8-deg and 13.5-deg data. The two sets of data fit fairly closely the same curve throughout the range of w. This indicates that the theory of final-state interactions provides a reasonable explanation for the data.

So far we have not considered the 15.7-deg data (see Fig. 6), and we have not included the Bose effect.

When all factors are included—the Bose effect (Sec. IVC), the He^3 form factor effects (Sec. IVD), and the π-π enhancement factor—we can fit all the data with the same value of the π-π scattering length. There are some reservations, however. In our method of analysis, we just multiply all these factors together, assuming them to be independent. This could possibly be a bad assumption. In addition, we have chosen arbitrarily one of two types of phase-space calculation. Also we have assumed that the $(1+\cos^2\theta^*)$ angular distribution in the reaction $p+p\rightarrow d+2\pi$ is independent of the relative π-π energy, whereas it is experimentally determined as an average over the possible values of the π-π energy. Fortunately, our results are rather insensitive to the relative amounts of isotropic and $\cos^2\theta^*$ dependence. Finally, the Bose symmetrization introduces a second parameter. Since the shape of the symmetrization factor is somewhat similar to that of the π-π enhancement factor, we can vary the Bose radius and the scattering length together to achieve equally acceptable fits to the data. One can guess at a value for the Bose radius—a physically reasonable range might be between $\frac{1}{2}$ and $2\hbar/\mu c$. In fact these limits are similar to those found from our data-fitting procedure.

VI. RESULTS

In practice, the phase space was multiplied by the He^3 form-factor effects, by the symmetrization factor and by the π-π enhancement factor for a set of values of the Bose parameter R and the scattering length a_{s0}. The experimental resolution was folded in on the IBM 7090 computer and the result fitted to the data. A goodness-of-fit parameter M and a normalizing factor were printed out. All three sets of data gave similar results; however, the 15.7-deg data are much less sensitive to the value of the scattering length. Figure 13 shows goodness-of-fit contours summed over the three sets of data. The $M=140$ contour corresponds to what we feel is the limit of an acceptable fit, both from the computer fitting procedure and visual comparison with the data. The lowest value of M obtained was $M=124$. From Fig. 13 we can see that any value of the scattering length in the range 1 to 3 $\hbar/\mu c$ is acceptable, the required value decreasing as the Bose radius is increased up to $2\hbar/\mu c$. For values of the Bose radius greater than $2\hbar/\mu c$, the Bose factor loses its effect, and the scattering length increases again. In fact, for $R=\infty$ we have the same situation as with $R=0$. We consider these larger values of $R(>2\hbar/\mu c)$ to be physically unreasonable.

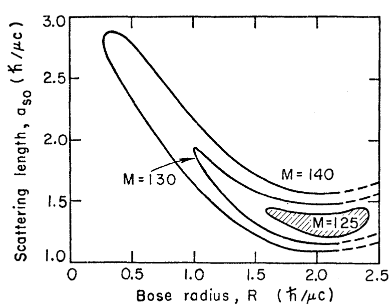
Fits to the three sets of experimental data for various values of the parameters are shown in Fig. 14.

Until it is possible to determine more exactly the Bose effect in an independent way, either theoretically or experimentally, we find the $I=0$ π-π scattering length to be $(2\pm1)\hbar/\mu c$.

VII. DISCUSSION

The scattering length determined above corresponds to an attractive interaction. Watson[11] has shown that a repulsive interaction would have very little effect in an experiment such as ours. Apart from the considerations of the preceding section, the value of the scattering length determined from these experiments depends upon the energy dependence of the phase shift assumed. We have taken the energy dependence given in Eq. (5) which is derived from the work of Chew and Mandelstam. However, as Hamilton *et al.*[16] point out, other authors have derived or used different expressions, and

FIG. 13. Goodness-of-fit contours summed over the three sets of data (105 experimental points). The region inside the $M=140$ contour corresponds to acceptable fits.



[16] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

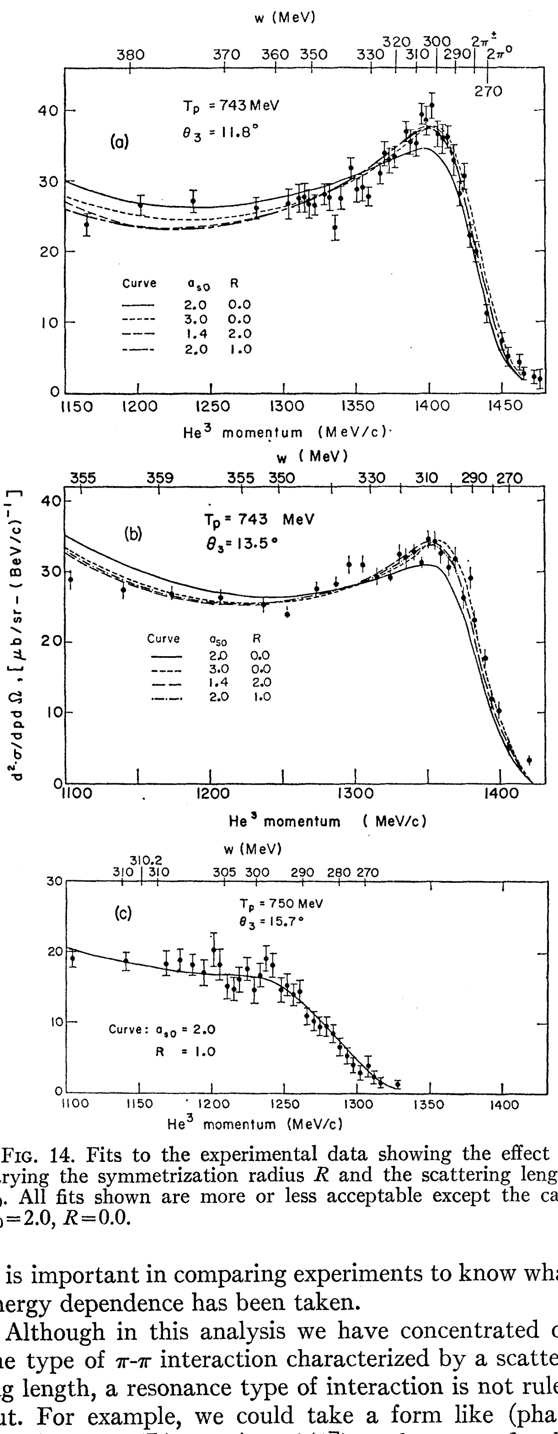


FIG. 14. Fits to the experimental data showing the effect of varying the symmetrization radius R and the scattering length a_{s0}. All fits shown are more or less acceptable except the case $a_{s0}=2.0$, $R=0.0$.

it is important in comparing experiments to know what energy dependence has been taken.

Although in this analysis we have concentrated on the type of π-π interaction characterized by a scattering length, a resonance type of interaction is not ruled out. For example, we could take a form like (phase space)$\times\{1+c/[(w-w_R)^2+\Gamma^2/4]\}$ and try to fit the data by adjusting the three parameters c, w_R, and Γ. One possible solution has $c\gg1$ and $w_R\lesssim 2m_\pi$. That is, the curves of Fig. 12 are similar in shape to the high-energy tail of a Breit-Wigner formula. In any case, w_R cannot be more than about 20 MeV above threshold.

In the next section we compare our conclusions about the $I=0$ π-π interaction with other experiments and theory. In general, the comparisons are arranged in order of increasing amount of theory between experiment and conclusions.

VIII. COMPARISON WITH OTHER EXPERIMENTS AND THEORY

We have shown how our experiments may be analyzed in terms of a strong $I=0$ S-wave pion-pion attraction. This attraction may be thought of as a "virtual state" similar to the well-known singlet n-p S-wave "virtual state." Let us now summarize methods of observing or calculating the $I=0$ S-wave π-π interaction. We shall use the scattering length as a measure of the strength of the interaction and shall compare our work with other results where possible.

A. Final-State Interaction in Pion-Production Experiments

This is the method we have used. Other possible experiments are $\pi+N\rightarrow 2\pi+N$ and $\gamma+N\rightarrow 2\pi+N$. Two other groups at Berkeley have looked at the reactions $\pi^-+p\rightarrow\pi^++\pi^-+n$ and $\pi^-+p\rightarrow\pi^0+\pi^0+n$ over a range of energies.[17,18] Both groups do not observe any marked effect similar to that seen in our experiments. Instead, they observe a peaking of the neutron-energy distributions corresponding to the maximum kinematically possible relative π-π energy. To our knowledge, this effect is so far unexplained and appears to be so strong as to obliterate the effects of a large scattering length. On the other hand, the energy dependence of the pion-production cross sections in the various final charge states can be explained in terms of a strong S-wave π-π interaction.[17,19] Both modes ($\pi^+\pi^-n$ and $\pi^0\pi^0n$) in which the $I=0$ S-wave π-π state is available rise rapidly from threshold, whereas the $\pi^-\pi^0p$ mode remains low. As the $I=1$ P-wave π-π interaction becomes significant, the $\pi^-\pi^0p$ channel begins to rise, and the $\pi^+\pi^-n$ mode continues to increase. The $\pi^0\pi^0n$ channel, without access to the $I=1$ π-π state, levels off as the S-wave π-π interaction falls off.

Thus, there appears to be a paradox. Until we understand the mechanism of these reactions, we feel that they neither support nor contradict our conclusions. Also, it is possible that the Bose effect might be less for a $N\pi\pi$ final state than for a He^3 $\pi\pi$ final state. This would make the effect of the scattering length more difficult to observe in the $N\pi\pi$ case.

There are some other experiments on pion pro-

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duction[20] in which the data appear to support our conclusions.

B. Final-State Interaction in Decay Processes

The processes $K^+ \to \pi^+ + \pi^+ + \pi^-$ and $K^+ \to \pi^+ + \pi^0 + \pi^0$ have been studied for some time to obtain information on a_{s2} and a_{s0}, the $I=2$ and $I=0$ π-π scattering lengths. The original work of Khuri and Treiman[21] gave $a_{s2} - a_{s0} \simeq 0.7\hbar/\mu c$. However, recent work by Bég and DeCelles shows that the situation is very different if one takes into account the effects of the known P-wave π-π interaction.[22] With $a_{s0} \simeq 2\hbar/\mu c$, they obtain reasonable agreement with the data on τ and τ' decays.

C. Pion-Pion Scattering Using the One-Pion Exchange Model

Chew and Low[23] have shown that, in the reaction $\pi + N \to 2\pi + N$, the region of low-momentum transfers to the struck nucleon is likely to be dominated by the pole associated with the exchange of a single pion between the incoming pion and nucleon. They give a method by which one may extrapolate from the physical region to this pole and determine the pion-pion scattering cross section. These experiments are difficult because that part of the physical region dominated by the one-pion pole may be small with respect to the extrapolation distance. It is also difficult to show that the region one has measured is in fact dominated by the pole. However, experiments of this type have been successful in verifying the existence of the $I=1$ P-wave π-π resonance.[24] Unfortunately there is not sufficient data at low relative π-π energies to give any information on a_{s0}.

Ceolin and Stroffolini have assumed that the *total* cross section for $\pi^- + p \to \pi^- + \pi^+ + n$ is given by the one-pion exchange for incident pion energies from threshold to 317 MeV.[25] They take $a_{s2} = \pm 0.54\hbar/\mu c$ to check the interference between $I=0$ and $I=2$ and calculate total cross sections for values of a_{s0} between 1.0 and $2.8\hbar/\mu c$. This analysis does not seem to be self-consistent. The data below 280 MeV require $a_{s0} < 1.0$, while data at 290 and 317 MeV require $a_{s0} \simeq 1.3$, and at 380 MeV $a_{s0} \approx 2.6$.

D. Other Models of Pion Production

Schnitzer has used a model based on static theory to extract π-π scattering lengths from total cross sections and angular distributions of $\pi + N \to 2\pi + N$.[26] He obtains two solutions for the scattering lengths $(a_0; a_1; a_2)$: (0.5, 0.07, 0.16) and (0.65, 0.07, −0.14). Hamilton *et al.* have pointed out shortcomings in this analysis.[16] In particular they feel that Schnitzer's analysis tends to underestimate a_0.

Anselm and Gribov have shown how the energy distribution of the secondary particles in, for example, the reaction $\pi^- + p \to \pi^- + \pi^+ + n$ near threshold may be used to extract the charge-exchange cross section $\pi^- + \pi^+ \to \pi^0 + \pi^0$.[27] So far this theory has had only a very preliminary test.[28]

E. Threshold Anomalies

This method is based on the observation of a cusp or rounded step in the cross section for a reaction with three particles in the final state.[29,30] There must be (at least) two different reactions with three particles in the final state, one of the particles must be common to reactions, and the phenomenon must be analyzed by keeping the total energy fixed. An example is the case of K^+ decay: an anomaly should occur in the energy spectrum of the π^+ from $K^+ \to \pi^+ + \pi^0 + \pi^0$ at the threshold for $K^+ \to \pi^+ + \pi^+ + \pi^-$. This anomaly would then give information about $\sigma(\pi^+ + \pi^- \to \pi^0 + \pi^0)$.[31] However, these threshold effects are expected to be small and have not yet been observed.

F. Pion-Nucleon Scattering

Pion-pion interactions must certainly affect pion-nucleon scattering, and any complete theory must certainly include them. Recently some progress along these lines has been made by several theoreticians. Ishida *et al.*[32] and Efremov *et al.*[33] have used dispersion relations for pion-nucleon scattering including pion-pion effects. The $I=0$ S-wave π-π phase shift enters with a large factor, while the $I=1$, P-wave phase shift does not appear to be so important. Both sets of workers find the $I=0$, S-wave interaction to be strongly attractive with $a_{s0} \sim 1\hbar/\mu c$. In recent, more complete cal-

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culations Hamilton *et al.* obtain $0.6 \leqslant a_{s0} \leqslant 2.0\hbar/\mu c$.[16] Comparing their results with the π-π equations of Chew and Mandelstam, they can limit the uncertainties in a_{s0} and obtain $a_{s0} = (1.3 \pm 0.4)\hbar/\mu c$.

G. Nucleon-Nucleon Scattering

The dispersion-relation approach may also be applied to nucleon-nucleon scattering. Cziffra *et al.* incorporated the one-pion pole into an analysis of 300-MeV p-p scattering.[34] They were able to show that the higher partial waves are adequately represented for the acceptable phase shift solutions by the single-pion-exchange pole. Wong recently extended the calculation to include multipion exchanges.[35] The inclusion of the ρ and ω states explains a large part of the force but does not give enough medium-range attraction. An $I=0$ $J=0$ π-π pair seems to be needed, although, at the moment, the width and effective energy of the state are not well determined.

H. Solutions of the π-π Equations

Recently solutions of the Chew-Mandelstam equations for π-π scattering have been derived by several authors.[36–38] Although the situation is not yet clear enough to make quantitative comparison, all calculations consistent with a P-wave π-π resonance give a fairly strong $I=0$ S-wave π-π attraction, in agreement with our result.

I. High-Energy Cross Sections and Regge Poles

Chew and Frautschi[39] have pointed out that the strong $I=0$ attraction in the π-π system should be associated with a Regge pole[40] commonly called[41–43] the ABC pole, with α_{ABC} its position in the complex angular momentum plane. Barut has investigated the Regge trajectory of such a "virtual state,"[42] and Udgaonkar has employed the trajectory in the analysis of high-energy cross sections.[41] The currently available total cross-section data at high energies is consistent with our assignment of $a_{s0} \approx 2\hbar/\mu c$.

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